Topic 10. Analysis of variance model for Fixed, Random and Mixed effects

References: ST&DT: Topic 7.5 p.152-153, Topic 9.9 p. 225-227, Topic 15.5 379-384, rules for expected MS ST&D page 381 replaced by Chapter 8, Montgomery D.C. 1991. "Design and analysis of experiments" 3rd ed. John Wiley & Sons. Good discussion on SAS System for Linear Models, 3rd ed. pages 191-198.

10. 1. Introduction

The experiments discussed in previous chapters have primarily dealt with situations where the experimenter is concerned with specific treatment comparisons. There are other types of experiments, however, where the investigator is concerned with the variability of treatments rather than comparisons among them. The purpose of this chapter is to introduce analysis of variance models appropriate to various experimental objectives.

10. 2. Fixed and Random Models in One-way Classification Experiments

10. 2. 1. Fixed-effects model

Typical comparative experiments concern the comparison of several treatment-means, for example, four different forms of nitrogen fertilizer are compared in a CRD with five replications. The form of the analysis of variance is:

Source	df
Treatment	3
Within Treatment	16

The statistical model for this experiment is: $Y_{ij} = \mu + \alpha_i + \varepsilon_{ij}$

In this experiment, there are four **specific treatments** involved. In using them, the experimenter has no thought for any other N-fertilizers. That is, the experimenter's attention is fixed upon these four fertilizer treatments and no other.

The treatment effects are called fixed effects when each α_i will be the same in each replication of the experiment. Even if the experiment is not replicated, one can think of a probabilistic model in which there are very many possible outcomes and the actual experiment selects one of them. In this context to say that the effects are fixed means that in each possible outcome the α_i's have the same value. Also, the α_i's must add up to 0 since the mean over all the treatments is μ. The model is called the **fixed-effects model** or **Model I**. In this experiment the ε_{ij} on each occasion would be one random sample from a population of errors with a distribution of 0 mean and a variance, σ^2.

10. 2. 2. Random-effects model

- The treatments are a ***random sample*** from a larger population of treatments for which the mean is zero and the variance is σ^2_t

- The **objective** of the researcher ***is to extend the conclusions*** based on the sample of treatments ***to ALL treatments*** in the population
- Here the **treatment effects** are **random variables** ($\mathbf{s}_i$) and ***knowledge about the particular* ones** investigated is relatively ***useless***
- Model $Y_{ij} = \mu + \mathbf{s}_i + \varepsilon_{ij}$
- where $\mathbf{s}_i$ will be **different** if the experiment is repeated

$\sum \mathbf{s_i} \neq \mathbf{0}$ $\mathbf{s_i}$ **are N(0,** $\sigma^2_{\mathbf{s}}$**)** H_0: is $s^2_{\mathbf{s}}=0$ vs. H_A: $s^2_{\mathbf{s}} \neq 0$

- When the null hypothesis is false, there will be an additional component of variance equal to $\mathbf{r\sigma^2_s}$.
- The researcher wants to test the presence and estimate the magnitude of the added variance component among groups: $\sigma^2_{\mathbf{s}}$.
- For One Way ANOVA, the computation is the same for the **fixed** and **random** Models. However, the **objectives** and the **conclusions** are different. The computations following the initial significance test are also different.

Example

In an animal breeding experiment to estimate sire combinability and breed variance, several sires are randomly selected from a population and each sire is mated with several dams. The average body weights of newborn animals are recorded as observations. Let us assume four sires were chosen and each was mated with 6 dams. The experimental model is: $Y_{ij} = \mu + s_i + \varepsilon_{ij}$

where s_i is the i^{th} sire breeding effect (the difference between the i^{th} sire and the overall mean). The other terms are the same as the fixed-effects model. In this case, the four sires used in the study are of no specific interest; they are merely a sample from a population of sires of the breed. If the experiment is to be repeated, very likely, another set of sires will be used.

The s_i's represent a random sample from a population of s's with mean 0 (since the mean response of the whole population is still μ and the variance is σ^2_s. In general, however the s_i's will be different in each replication of the model and will not add up to zero in a particular replication. This is the **random-effects model** and is sometimes called **Model II**.

If we let μ_i to be the mean breeding value of the i^{th} sire, and μ is the overall breeding value of all sires in the population, then for any given sample.

$$\Sigma s_i = \Sigma(\mu - \mu_i) \neq 0$$

unless the summation covers the entire population. The distribution of s_i has a mean of 0 and variance of σ^2_s.

Sometimes the determination of whether an effect is fixed or random is not obvious. Examples are laboratories or technicians in a comparative study, or years in a multiple-year trial. These factors can be fixed or random depending on the objective of the study, the intended inference to be made, and the sampling process as to how the factors are selected.

10. 2. 3. Differences between fixed and random-effects model

Although the mathematical models of the above two types of experiments are similar for a CRD, there are some fundamental differences:

1) **The objectives are different**. In the fertilizer experiment each fertilizer is of specific importance. They are not a random sample from a population of fertilizer. The purpose of the study is to compare these treatments. For the breeding study, however, the objective of the study is to estimate the combinability of a breed. The sires used in the study are merely **a sample** from which inferences are to be made concerning the **population**.

   The purpose of a fixed model is to test the hypothesis that the treatment effects are the same; the purpose of a random model is to estimate σ^2_s and σ^2_ε, the variance components.

2) **The sample procedures are different**. In the fixed effects model experiment the treatments are not randomized but are selected purposefully by the experiment. In the random effects model the treatments are randomly selected and the variance in the population of treatments contributes to the total sum of squares. If the first experiment is repeated, the four fertilizers must be used again and only the random errors are changed from experiment to experiment, i.e., α_i's are assumed to be constants and do not change, only ε_{ij}'s change. In the second experiment, the four sires most likely will be changed from experiment to experiment. Thus not only the errors are changeable but the sire effect, s_i, are changeable.

3) **The expected sums of the effects are different**. For any given sample, in the fixed –effect model $\Sigma\alpha_i = \Sigma(\mu-\mu_i)= 0$. For any given sample, in the random–effect model $\Sigma s_i = \Sigma(\mu-\mu_i) \neq 0$

4) **The expected variances are different**

   For model I:

   Var. of Y_{ij} = variance of μ + variance of α_i + variance of ε_{ij}= variance of ε_{ij} = σ^2_ε

   For model II:

   Var. of Y_{ij} = variance of μ + variance of s_i + variance of ε_{ij}= variance of s_i + variance of ε_{ij}= $\sigma^2_s + \sigma^2_e$ (σ^2_s and σ^2_ε are called **variance components)**

   The expected mean squares for these models are shown in Table 1.

Table 1. Expected mean of one-way classification experiments.

			EMS	
Source	df	MS	Fixed Model	Random Model
Treatment	a-1	MST	$\sigma^2_\varepsilon + r\,\Sigma\alpha^2_I/(a-1)$	$\sigma^2_\varepsilon + r\,\sigma^2_s$
Exp. error	a (r-1)	MSE	σ^2_ε	σ^2_ε

Suppose five cuts of meat are taken from each of three pigs all from the same breed and the fat content is measured in each cut. The goal is to assess animal-to-animal and within –animal variation. The particular three pigs selected are not of interest. This is an example with random effects, three treatment groups, and five replicates per group.

Suppose the treatment mean square is 80 and the error mean square is 20. The F ratio 80/20= 4 with 2 and 12 degrees of freedom is larger than the critical value and hence significant at the 5% level. We conclude that $\sigma^2_s>0$, but how big is it?

Table 1 implies that 80 is an estimate of $\sigma^2_\varepsilon + r\,\sigma^2_s = \sigma^2_\varepsilon + 5\,\sigma^2_s$ and 20 is an estimate of σ^2_ε. The difference 80 –20=60 is an estimate of $5\,\sigma^2_s$, so 60/5 = 12 is an estimate of σ^2_s, the variance component due to animal-to-animal differences

10. 3. Two-way Classification Experiments

In the single factor case the model must specify either fixed or random effects. In the multifactor case these two possibilities are joined by a third: the mixed effect model in which some factors are fixed and some are random. An examination of this model is useful because it provides a good way to contrast fixed and random effects.

Suppose an experimenter conducts a field study, which includes several varieties tested at a number of locations. In each location a completely randomized design was used with each variety replicated in r plots. Let Y_{ijk} represent the plot yield of the k^{th} plot of the i^{th} variety at the j^{th} location. Then we have:

$$Y_{ijk} = \mu + \alpha_i + ß_j + (\alpha ß)_{ij} + \varepsilon_{ijk}$$

where μ is an overall mean, α_i is the effect of the i^{th} variety. $ß_j$ is the effect on yield of the j^{th} location, $(\alpha ß)_{ij}$ is the interaction and ε_{ijk} is the usual random error term with mean 0 and variance σ^2_ε. There are several possible models for this study:

10. 3. 1. Fixed Effects Model

If the experimenter is only interested in particular in the varieties used at the specific locations selected and conclusions are not to be generalized to other varieties or locations, then the model is called a fixed-effects model. In this case:

1) μ is the mean of the variety means in the study.
2) $\alpha_i = \mu_{i.} - \mu$ is the true effect of the i^{th} variety and $\Sigma\alpha_i = 0$.
3) $ß_j = \mu_{.j} - \mu$ is the true effect of the j^{th} location and $\Sigma ß_j = 0$.
4) $(\alpha ß)_{ij}$ = is the specific interaction effect due to the i^{th} variety and the j^{th} location

The variance of Y_{ijk} is σ^2_ε. In this model, the experimenter is interested in estimating and testing hypotheses about α_i, β_j, and $(\alpha\beta)_{ij}$.

10. 3. 2. Random effects model

Suppose the varieties are randomly chosen from a population of varieties and the investigator is interested in variation among varietal yield from a population of varieties. Similarly, the locations are randomly selected from numerous possible testing sites and the specific locations in the trial are of no particular interest. In this situation,

1) μ is the overall mean of all conceivable varieties at all possible locations.
2) α_i is the deviation of the average yield of the i^{th} variety from μ. The α_i's can be considered as a random effect from a population with mean 0 and variance σ^2_α.
3) β_j is the deviation of the average yield of the j^{th} location from μ, and is considered a random effect from a population of location effects with mean 0 and variance σ^2_β .
4) $(\alpha\beta)_{ij}$ represents the random interaction effect and is considered to be from a population of all possible interaction effects of varieties and locations with a mean 0 and variance $\sigma^2_{\alpha\beta}$.

The variance Y_{ijk} in this model is $\sigma^2_\alpha + \sigma^2_\beta + \sigma^2_{\alpha\beta} + \sigma^2_\varepsilon$. In this model, the experimenter is interested in estimating and testing hypotheses about σ^2_α, σ^2_β, $\sigma^2_{\alpha\beta}$.

10. 3. 3. Mixed-effects model

Suppose the varieties are specifically chosen from comparisons but the locations are randomly selected from many possible locations to examine the consistency of the varietal yields under various environmental conditions. In this case,

1) μ is the overall average of these particular varieties at all possible locations.
2) α_i is the i^{th} varietal yield relative to the overall mean and $\Sigma\alpha_i = 0$.
3) β_j is the random effect of the j^{th} location sampled from a population of mean 0 and variance σ^2_β.
4) $(\alpha\beta)_{ij}$ is the interaction effect of the i^{th} variety in j^{th} location. Since locations are random, it is usually considered as a random effect with mean 0 and variance $\sigma^2_{\alpha\beta}$.

The variance of Y_{ijk} is $\sigma^2_\beta + \sigma^2_{\alpha\beta} + \sigma_\varepsilon^2$. In this model, the experimenter is interested in making inferences about α_i, σ^2_β and $\sigma^2_{\alpha\beta}$.

10. 3. 4. Expected mean squares and F tests.

In all the examples of fixed models discussed in previous topics, the error term is always the last line of the partitioning of the sum of squares. The proper tool for determining appropriate error variances for more complex situations is the set of **expected mean squares**. Expected mean squares are algebraic expressions specifying the functions of the model parameters that are estimated by the mean squares resulting from partitioning the sum of squares. Generally, these expected mean squares are linear functions of elements that represent:

- The error variances
- Functions of variances of random effects
- Functions of sums of squares and products (quadratic forms) of fixed effects

Table 2. Expected mean squares of two-way classification experiments with a varieties, b locations, and r replications.

Source	df	MS	Fixed Model	Random Model	Mixed Model Loc.=Rand. Var.=Fixed
Loc.	b-1	MSL	$\sigma^2_\varepsilon + ar\Sigma\beta^2/(b-1)$	$\sigma^2_\varepsilon + r\sigma^2_{\alpha\beta} + ra\sigma^2_\beta$	$\sigma^2_\varepsilon + r\sigma^2_{\alpha\beta} + ra\sigma^2_\beta$
Variety	a-1	MSV	$\sigma^2_\varepsilon + br\Sigma\alpha^2/(a-1)$	$\sigma^2_\varepsilon + r\sigma^2_{\alpha\beta} + rb\sigma^2_\alpha$	$\sigma^2_\varepsilon + r\sigma^2_{\alpha\beta} + br\Sigma\alpha^2/(a-1)$
L * V	(b-1)(a-1)	MSLV	$\sigma^2_\varepsilon + r\,\Sigma\Sigma(\alpha\beta)^2/(b-1)(a-1)$	$\sigma^2_\varepsilon + r\sigma^2_{\alpha\beta}$	$\sigma^2_\varepsilon + r\sigma^2_{\alpha\beta}$
Exp. err.	ba(r-1)	MSE	σ^2_ε	σ^2_ε	σ^2_ε

There is **controversy** among statisticians in relation to the inclusion of interactions between fixed and random factors in the expected mean squares of the random factors. Steel, Torrie and Dickey exclude the $r\sigma^2_{\alpha\beta}$ term from the expected mean squares of the random variable (see the term in parenthesis in the expected MS location in Table 2). However, Hocking (J of the Am Statistical Association 1975 70:706-712) pointed out that exclusion of $\sigma^2_{\alpha\beta}$ is inconsistent with results commonly reported for the unbalanced case. Hocking's criteria is also followed by SAS and in this syllabus, that is we will include the interactions between fixed and random factors in the expected mean squares of random factors in a mixed model. Note also that the expected MS for interactions in the mixed model in ST&D p 380 and p 226 include an additional correction factor of the form x/(x-1). This factor is not used in most statistic books including SAS. ST&D also point this out on p 382, first paragraph.

Based on the expected mean squares an appropriate denominator mean square can be found to form an F test concerning location, variety and interaction. Appropriate F test for the models are shown in Table 3.

Table 3. Appropriate F tests for various models.

		Fixed	Random	Mixed
Location	b-1	MSL/**MSE**	MSL/MSLV	MSL/ MSLV
Variety	a-1	MSV/**MSE**	MSV/MSLV	MSV/MSLV
Interaction	(b-1)(a-1)	MSLV/**MSE**	MSLV/**MSE**	MSLV/**MSE**

Note that the appropriate F tests change, depending on the model.

The underlying principle of an F test on a set of fixed-effect parameters is that the expected mean square for the *denominator* contains a linear function of variances of random effects, whereas the expected mean square for the *numerator* contains the same function of these variances **plus** a quadratic form of the parameters being tested.

For **fixed models**, all mean squares estimate the residual error variance plus a quadratic form of the parameters in question. Hence, the proper denominator for all tests is the error term. That's why expected mean squares are usually not required for fixed models. For **random and mixed models** it is essential to estimate the expected mean squares to select the appropriate F test. In the random model you want to extend your conclusion to the complete population of treatments (e.g. locations in the Mixed model above). Therefore it makes sense to divide by the MS interaction: your variety effect to be valid across all locations should be larger than the genotype x environment interactions.

In Topic 9.7.2, it was pointed out that a significant interaction in the **fixed model** may cause us to lose interest in tests of hypothesis concerning main effects and to become interested in tests for simple effects. However, for a **mixed model** we may not be at all interested in simple effects for a fixed effect, since they will be measured at randomly selected levels of another factor. Instead, we will be interested in the main effects even when there is an interaction between the random and the fixed effect.

10. 3. 4. 1. Rules for Determining Expected Mean Squares

An important part of any experimental design problem is conducting the analysis of variance. This involves determining the sum of squares for each component in the model and the number of degrees of freedom associated with each sum of squares. Then, to construct appropriate test statistics, the expected mean squares must be determined. In complex design situations, particularly those involving random or mixed models, it is frequently helpful to have a formal procedure for this process.

This set of rules is appropriate to calculate the number of degrees of freedom, and the expected mean squares for any balanced factorial, nested, or nested factorial, experiment. (Note that partially balanced arrangements, such as Latin squares and incomplete block designs, are specifically excluded.)

The test statistic is a ratio of mean squares that is chosen such that the expected value of the *numerator* mean square differs from the expected value of the *denominator* mean square only by the variance component or the fixed factor in which we are interested.

The two-factor factorial model is used to illustrate the application of the rules.

RULE 1. The error term in the model, $\varepsilon_{ij...m}$, is written as $\varepsilon_{(ij...)m}$, where the subscript m denotes the replication subscript. For the two-factor model, this rule implies that ε_{ijk} becomes $\varepsilon_{(ij)k}$.

RULE 2. In addition to the error term the model contains all the main effects and any interactions that the experimenter assumes exist.

RULE 3. For each term in the model, divide the subscripts into three classes:

a) **Live**: those subscripts that are present in the term and are not in parentheses.

b) **Dead**: those subscripts that are present in the term and are in parentheses (usually subscripts of nested factors).
c) **Absent**: those subscripts that are present in the model but not in that particular term.

For example, in $(\alpha\beta)_{ij}$ i and j are live and k is absent, and in $\varepsilon_{(ij)\,k}$ k is live and i and j are dead.

RULE 4. **Degrees of freedom**. The number of degrees of freedom for any term in the model is the product of the number of levels associated with each dead subscript and the number of levels minus 1 associated with each live subscript.

For example: df $(\alpha\beta)_{ij}$= (a-1)(b-1); df $\varepsilon_{(ij)\,k}$=ab(r-1)

RULE 5. Each effect has either a variance component (random effect) or a fixed factor (fixed effect) associated with it. If an interaction contains at least one random effect, the entire interaction is considered as random. A variance component has Greek letters as subscripts to identify the particular random effect. Thus, in a two-factor mixed model with factor A fixed and factor B random, the variance component for B is σ^2_β and the variance component for AB is $\sigma^2_{\alpha\beta}$. A fixed effect is always represented by the sum of squares of the model components associated with that factor divided by its degrees of freedom. In our example, the effect of A is:

$$\frac{\sum_{i=1}^{a}\alpha_i^2}{a-1}$$

RULE 6. **Expected Mean Squares**. To obtain the expected mean squares, prepare the following table. There is a row for each model component (mean square) and a column for each subscript. Over each subscript, write the number of levels of the factor associated with that subscript and whether the factor is fixed (F) or random (R). Replicates are always considered to be random.

(a) In each row, write 1 if one of the dead subscript in the row component matches the subscript in the column:

Fixed or Random Number of levels Factor	F a i	R b j	R r k
α_i			
β_j			
$(\alpha\beta)_{ij}$			
$\varepsilon_{(ij)\,k}$	1	1	

(b) In each row, if any of the subscripts on the row component match the subscript in the column, write 0 if the column is headed by a fixed factor and 1 if the column is headed by a random factor. For interactions including at least 1 random factor write 1 in the columns that match the row subscript, independently of the nature (fixed or random) of the column:

Fixed or Random Number of levels Factor	F a i	R b j	R r k
α_i	0		
β_j		1	
$(\alpha\beta)_{ij}$	**1**	1	
$\varepsilon_{(ij)k}$	1	1	1

Note the inclusion of a **1** in the a column for the $(\alpha\beta)_{ij}$. This will determine the presence of $r\sigma^2_{\alpha\beta}$ in the expected MS of the random Factor β.

(c) In the remaining empty row positions, write the number of levels shown above the column heading:

Fixed or Random Number of levels Factor	F a i	R b j	R r k
α_i	0	b	r
β_j	a	1	r
$(\alpha\beta)_{ij}$	**1**	1	r
$\varepsilon_{(ij)k}$	1	1	1

(d) To obtain the expected mean square for any model component, first cover all columns headed by live subscripts on that component. Then, in each row that contains at least the same subscripts as those on the component being considered, take the product of the visible numbers and multiply by the appropriate fixed or random factor from Rule 5. The sum of these quantities is the expected mean of the model component being considered.

To find $E(MS_A)$, for example, cover column i. The product of the visible numbers in the rows that contain at least subscript i are br (row 1), r (row 3), and 1 (row 4). Note that i is missing in row 2.

The expected mean squares derived using these rules for the two-way mixed model are:

Fixed or Random Number of levels Factor	F a i	R b j	R r k	Expected Mean Squares
α_i	0	b	r	$\sigma^2_\varepsilon + r\sigma^2_{\alpha\beta} + br\Sigma\alpha^2/(a-1)$
β_j	a	1	r	$\sigma^2_\varepsilon + \mathbf{r\sigma^2_{\alpha\beta}} + ar\sigma^2_\beta$ (ST&D excludes $r\sigma^2_{\alpha\beta}$)
$(\alpha\beta)_{ij}$	**1**	1	r	$\sigma^2_\varepsilon + r\sigma^2_{\alpha\beta}$
$\varepsilon_{(ij)k}$	1	1	1	σ^2_ε

10. 4. Expected Mean squares for three-way ANOVA

Consider a three-factor factorial experiment with a levels of factor A, b levels of factor B, c levels of factor C, and r replicates. The analysis of this design assuming that A is fixed and B and C are random is given below. The appropriate statistical model is:

$$Y_{ijkl} = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk} + \varepsilon_{ijkl}$$

Fixed or Random Number of levels Factor	F a i	R b j	R c k	R r l	**Expected Mean Squares**	**F**
α_i	0	b	c	r	$\sigma^2_\varepsilon + r\sigma^2_{\alpha\beta\gamma} + br\sigma^2_{\alpha\gamma} + cr\sigma^2_{\alpha\beta} + bcr\Sigma\alpha^2/(a-1)$	**?**
β_j	a	1	c	r	$\sigma^2_\varepsilon + \mathbf{r\sigma^2_{\alpha\beta\gamma}} + ar\sigma^2_{\beta\gamma} + \mathbf{cr\sigma^2_{\alpha\beta}} + acr\sigma^2_\beta$	**?**
γ_k	a	b	1	r	$\sigma^2_\varepsilon + \mathbf{r\sigma^2_{\alpha\beta\gamma}} + ar\sigma^2_{\beta\gamma} + \mathbf{br\sigma^2_{\alpha\gamma}} + abr\sigma^2_\gamma$	**?**
$(\alpha\beta)_{ij}$	**1**	1	c	r	$\sigma^2_\varepsilon + r\sigma^2_{\alpha\beta\gamma} + cr\sigma^2_{\alpha\beta}$	$MS_{\alpha\beta}/MS_{\alpha\beta\gamma}$
$(\alpha\gamma)_{ik}$	**1**	b	1	r	$\sigma^2_\varepsilon + r\sigma^2_{\alpha\beta\gamma} + br\sigma^2_{\alpha\gamma}$	$MS_{\alpha\gamma}/MS_{\alpha\beta\gamma}$
$(\beta\gamma)_{jk}$	a	1	1	r	$\sigma^2_\varepsilon + \mathbf{r\sigma^2_{\alpha\beta\gamma}} + ar\sigma^2_{\beta\gamma}$	$MS_{\beta\gamma}/MS_{\alpha\beta\gamma}$
$(\alpha\beta\gamma)_{ijk}$	**1**	1	1	r	$\sigma^2_\varepsilon + r\sigma^2_{\alpha\beta\gamma}$	$MS_{\alpha\beta\gamma}/MS_\varepsilon$
$\varepsilon_{(ijk)l}$	1	1	1	1	σ^2_ε	

The 1's indicated in bold would be zero in ST&D. The variances in bold would be absent in ST&D.

Identical results to those presented in the previous table can be obtained in SAS using the RANDOM statement in PROC GLM. When the TEST option in the RANDOM statement is specified, it determines what tests are appropriate, and provides F ratios and probabilities for these tests. However, a contrast is not corrected by the RANDOM statement, and there is no way provided for the user to specify a synthetic denominator for a contrast (SAS provides an alternative procedure for this purpose: PROC MIXED).

All random effects and all interactions including at least one random effect must be explicitly designated as random in the SAS RANDOM statement

SAS Program (A=fix B and C= Random)

model Y= A B C A*B A*C B*C A*B*C;
random B C A*B A*C B*C A*B*C;.

This RANDOM statement will generate the following output:

```
Class      Levels     Values
A               3     1 2 3
B               5     1 2 3 4 5
C               2     1 2
r               1
```

```
Source        Type III Expected Mean Square
A             Var(Error) + Var(A*B*C) + 5 Var(A*C) + 2 Var(A*B) + Q(A)
```

```
B           Var(Error) + Var(A*B*C) + 3 Var(B*C) + 2 Var(A*B) + 6 Var(B)
C           Var(Error) + Var(A*B*C) + 3 Var(B*C) + 5 Var(A*C) + 15 Var(C)
A*B         Var(Error) + Var(A*B*C) + 2 Var(A*B)
A*C         Var(Error) + Var(A*B*C) + 5 Var(A*C)
B*C         Var(Error) + Var(A*B*C) + 3 Var(B*C)
A*B*C       Var(Error) + Var(A*B*C)
```

Proper F tests for the two-factor and three-factor interactions and random main effects can be constructed using different MS from this table. However no exact test exists for the main effect A, B, or C. That is, if we wish to test the hypothesis $\Sigma\alpha^2/(a-1) = 0$, we cannot form a ratio of two expected mean squares such that the only term in the numerator that is not in the denominator is $bcr\Sigma\alpha^2/(a-1)$. The same problem occurs for the main effects in a three-way ANOVA with all random effects. However, it is likely that tests on the main effects are of central importance to the experimenter.

This problem is considered in the next section.

10. 5. Approximate F tests

In factorial experiments with three or more factors involving a random or mixed model and certain other, more complex designs, there are frequently no exact tests statistics for certain effects in the models. One possible solution to this dilemma is to assume that certain interactions are negligible.

Although this seems to be an attractive possibility, there must be something in the nature of the process-or some strong prior knowledge-in order to assume that one or more of the interactions are negligible. In general, this assumption is not easily made, nor should it be taken lightly. We should not eliminate certain interactions from the model without conclusive evidence that it is appropriate to do so.

A procedure advocate by some experimenters is to test the interactions first, then set at zero those interactions found to be insignificant, and then assume that these interactions are zero when testing other effects in the same experiment. Although sometimes done in practice, this procedure can be dangerous because any decision regarding an interaction is subject to both Type I and Type II errors.

If we cannot assume that certain interactions are negligible and we still need to make inferences about those effects for which exact tests do not exist, a procedure attributed to **Satterthwaite** (1946) can be employed. Satterthwaite's method uses **linear combinations of mean squares**, for example,

$$MS' = MS_r + \cdots + MS_S$$

and

$$MS'' = MS_U + \cdots + MS_V$$

where the mean squares are chosen so that no MS appear simultaneously in MS' and MS'', and E(MS') - E(MS″) is equal to the effect (the model parameter or variance component) considered in the null hypothesis. Then the test statistic would be

$$F = MS'/MS''$$

which is distributed approximately as $F_{p,q}$, where p and q are the ***effective degrees of freedom***.

$$\boldsymbol{p} = \frac{(MS_r + \cdots + MS_S)^2}{MS_r^2/df_r + \cdots + MS_S^2/df_S} \qquad \boldsymbol{q} = \frac{(MS_u + \cdots + MS_V)^2}{MS_U^2/df_U + \cdots + MS_V^2/df_V}$$

In p and q, df_i is the number of degrees of freedom associated with the mean square MS_i. There is no assurance that p and q will be integers, so it will be necessary to interpolate in the tables of the F distribution.

For example, in the three-factor mixed effects model discussed above, it is relatively easy to see that an appropriate test statistic for H_0: $\alpha_1 = ... = \alpha_t = 0$ would be

$$F = \frac{MS_A + MS_{ABC}}{MS_{AB} + MS_{AC}}$$

$$F = \frac{\sigma_\varepsilon^2 + r\sigma_{\alpha\beta\gamma}^2 + br\sigma_{\alpha\gamma}^2 + cr\sigma_{\alpha\beta}^2 + bcr\sum\frac{\alpha^2}{a-1} + \sigma_\varepsilon^2 + r\sigma_{\alpha\beta\gamma}^2}{\sigma_\varepsilon^2 + r\sigma_{\alpha\beta\gamma}^2 + br\sigma_{\alpha\gamma}^2 + cr\sigma_{\alpha\beta}^2 \quad + \quad \sigma_\varepsilon^2 + r\sigma_{\alpha\beta\gamma}^2}$$

Now, this is the F test for Factor A, the true effect of which is represented by the term

$$bcr\sum\frac{\alpha^2}{a-1}$$

that is the only one term that is not repeated in the numerator and the denominator

The degrees of freedom for F can be computed from the equations for ***p*** and ***q***. Since no mean square appears in both the numerator and denominator, the numerator and denominator are independent.

It is better to define additive linear combinations in the numerator and denominator than to subtract (e.g MS_A / (MS_{AB} + MS_{AC} - MS_{ABC})), because negative signs in such linear functions can lead to difficulties (ST&D pg. 380)

10.5.2. An additional example with nested effects

The rules given in this section are also applicable when a factorial experiment includes sampling, that is, when we have both a factorial and a nested experiment. Subscripts in parentheses indicate the position of the previous subscript in the hierarchy at which that component arises.

To illustrate the use of the rules with a nested classification, consider a two –way factorial A (fixed) x B (random) with C replications (random) nested in each A x B combination and D subsamples (random) per replication.

Fixed or Random Number of levels Factor	F a i	R b j	R c k	R d l	**Expected Mean Squares**	**F**
α_i	0	b	c	d	$\sigma^2_\varepsilon + d\sigma^2_{\gamma(\alpha\beta)} + cd\sigma^2_{\alpha\beta} + bcd\Sigma\alpha^2/(a-1)$	$MS_\alpha/MS_{\alpha\beta}$
β_j	A	1	c	d	$\sigma^2_\varepsilon + d\sigma^2_{\gamma(\alpha\beta)} + cd\sigma^2_{\alpha\beta} + acd\sigma^2_\beta$	$MS_\beta/MS_{\alpha\beta}$
$(\alpha\beta)_{ij}$	**1**	1	c	d	$\sigma^2_\varepsilon + d\sigma^2_{\gamma(\alpha\beta)} + cd\sigma^2_{\alpha\beta}$	$MS_{\alpha\beta}/MS_{\gamma(\alpha\beta)}$
$\gamma(\alpha\beta)_{k\,\mathbf{(ij)}}$	1	1	1	d	$\sigma^2_\varepsilon + d\sigma^2_{\gamma(\alpha\beta)}$	$MS_{\gamma(\alpha\beta)}/MS_\varepsilon$
$\varepsilon_{(ijk)l}$	1	1	1	1	σ^2_ε	

Note that the results are logic:

1) To test the interaction we used the MS of the factor nested in the interaction (as with any previous nested design). $MS_{\alpha\beta}/MS_{\gamma(\alpha\beta)}$
2) To test A or B we use the interaction $MS_{\alpha\beta}$ because we are trying to extend our conclusions across the universe of all treatment from which we extracted the B sample of treatments. Then the effect or A or B to be significantly larger, needs to be significantly larger than the interactions, which represent the differences in responses for A in the different B levels.